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| |  |  | | --- | --- | | http://www.regentsprep.org/Regents/math/algtrig/TRIG-IMAGES/TRIGLesson.jpg | **Definition of a Relation and a Function** [**Topic Index**](http://www.regentsprep.org/Regents/math/algtrig/ATP5/indexATP5.htm) **|** [**Algebra2/Trig Index**](http://www.regentsprep.org/Regents/math/algtrig/math-ALGTRIG.HTM) **|** [**Regents Exam Prep Center**](http://regentsprep.org) | |

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| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  | | --- | | **Relation:**  A relation is simply a set of ordered pairs. |     The first elements in the ordered pairs (the *x*-values), form the domain.  The second elements in the ordered pairs (the *y*-values), form the range.  Only the elements "used" by the relation constitute the range.   |  |  |  | | --- | --- | --- | | http://www.regentsprep.org/Regents/math/algtrig/ATP5/relationsets.gif | This mapping shows a **relation** from set A into set B. This relation consists of the ordered pairs (1,2), (3,2), (5,7), and (9,8).   |  | | --- | | •  The domain is the set {1, 3, 5, 9}. •  The range is the set {2, 7, 8}.    (Notice that 3, 5 and 6 are not part of the range.) •  The range is the dependent variable. | |  |  |  | | --- | --- | | The following are examples of relations.  Notice that a vertical line may intersect a relation in more than one location. | | | |  | | --- | | http://www.regentsprep.org/Regents/math/algtrig/ATP5/fixpic1.gif |   This set of 5 points is a relation. {(1,2), (2, 4), (3, 5), (2, 6), (1, -3)} Notice that vertical lines may intersect  more than one point at a time. | |  | | --- | | http://www.regentsprep.org/Regents/math/algtrig/ATP5/relationgraph3.gif |   This parabola is also a relation. Notice that a vertical line can  intersect this graph twice. |   If we impose the following rule on a relation, it becomes a function.   |  | | --- | | **Function:  A function is a set of ordered pairs in which each *x*-element has only ONE *y*-element associated with it.** |  |  |  |  | | --- | --- | --- | | The relations shown above are NOT functions because certain *x*-elements are paired with more than one unique *y*-element.  The first relation shown above can be altered to become a function by removing the ordered pairs where the *x*-coordinate is repeated.  It will not matter which "repeat" is removed.    **function:  {(1,2), (2,4), (3,5)}**  The graph at the right shows that a vertical line now intersects only ONE point in our new function. | |  | | --- | | http://www.regentsprep.org/Regents/math/algtrig/ATP5/fixpic2.gif | |  |  |  |  | | --- | --- | --- | | |  |  | | --- | --- | | **Vertical line test:** | each vertical line drawn through the graph will intersect a **function** in only one location. | |  |  |  | | --- | --- | | http://www.regentsprep.org/Regents/math/algtrig/ATP5/graph1.gif | http://www.regentsprep.org/Regents/math/algtrig/ATP5/graph2.gif | |

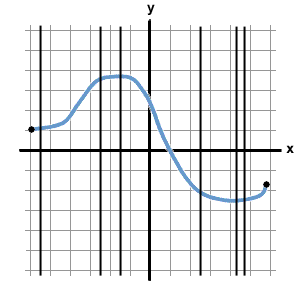
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| |  |  | | --- | --- | | http://www.regentsprep.org/Regents/math/ALGEBRA/ALG-IMAGES/math-ALGEBRAsmall.jpg | **Determining Relations and Functions** [**Topic Index**](http://www.regentsprep.org/Regents/math/ALGEBRA/AP3/indexAP3.htm) **|** [**Algebra Index**](http://www.regentsprep.org/Regents/math/ALGEBRA/math-ALGEBRA.htm) **|** [**Regents Exam Prep Center**](http://regentsprep.org) | |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  | | --- | | **Relation:**  A relation is simply a set of ordered pairs. |  |  |  |  | | --- | --- | --- | | A relation can be any set of ordered pairs. No special rules need apply. The following is an example of a relation: **relation**:  **{(1,2),(2, 4),(3, 5),(2, 6),(1, -3)}**  The graph at the right shows that a vertical line may intersect more than one point in a relation.  http://www.regentsprep.org/Regents/math/ALGEBRA/AP3/orangearrow.gif The graph of this orange arrow is also a relation. | |  | | --- | | http://www.regentsprep.org/Regents/math/ALGEBRA/AP3/fixpic1.gif | |   If we impose the following rule on a relation, it becomes a function.   |  | | --- | | **Function:**  A function is a set of ordered pairs in which each *x*-element has only ONE *y*-element associated with it. |  |  |  |  | | --- | --- | --- | | The relation shown above is NOT a function because the *x*-element 2 is paired with a *y*-element of 4 and **also** a *y*-element of 6.  Similarly, the *x-*element of 1 is paired with the *y-*elements of 2 **and** -3.  The relation above can be altered to become a function by removing the ordered pairs where the *x*-coordinate is used twice.   **function:  {(1,2), (2,4), (3,5)}**  The graph at the right shows that a vertical line intersects only ONE point in a function. This is called the **vertical line test** for functions. | |  | | --- | | http://www.regentsprep.org/Regents/math/ALGEBRA/AP3/fixpic2.gif | |  |  |  | | --- | --- | | **A function may not have two *y*-values assigned to the same *x*-value, such as {(2,4), (2,6)}. A function may, however, have two *x*-values assigned to the same *y*-value, such as {(2,4), (3,4)}.** | http://www.regentsprep.org/Regents/math/ALGEBRA/AP3/orangearrow.gif | |

**The Vertical and Horizontal Line Tests for Graphs**

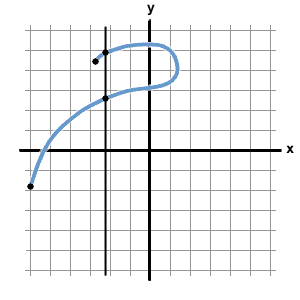
To determine whether *y* is a function of *x* , given a graph of a relation, use the following criterion: if every *vertical line* you can draw goes through only 1 point, *y* is a function of *x* . If you can draw a vertical line that goes through 2 points, *y* is not a function of *x* . This is called the vertical line test.

*Example 1*: In the following graph, *y* is a function of *x* :



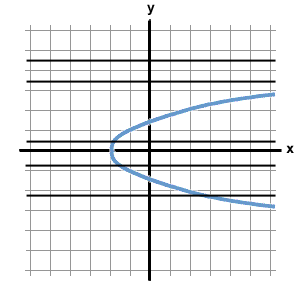
*Passes Vertical Line Test*

*Example 2*: In the following graph, *y* is not a function of *x* :

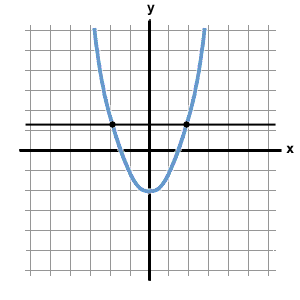
  
**Fails Vertical Line Test**

To determine whether *x* is a function of *y* , given a graph of a relation, use the following criterion: If every *horizontal line* you can draw passes through only 1 point, *x* is a function of *y* . If you can draw a horizontal line that passes through 2 points, *x* is not a function of *y* . This is called the horizontal line test.

*Example 1*: In the following graph, *x* is a function of *y* :

  
**Passes Horizontal Line Test**

*Example 2*: In the following graph, *x* is not a function of *y* :

  
**Fails Horizontal Line Test**

A ***relation*** is any association between elements of one set, called the ***domain*** or (less formally) the *set of inputs*, and another set, called the ***range*** or *set of outputs*. Some people mistakenly refer to the range as the *codomain*, but as we will see, that really means the *set of all possible outputs*—even values that the relation does not actually use.

For example, if the *domain* is a set Fruits = {apples, oranges, bananas} and the *codomain* is a set Flavors = {sweetness, tartness, bitterness}, the flavors of these fruits form a relation: we might say that apples are related to (or associated with) **both** sweetness and tartness, while oranges are related to tartness only and bananas to sweetness only. (We might disagree somewhat, but that is irrelevant to the topic of this book.) Notice that "bitterness", although it is one of the possible Flavors (codomain), is not really used for any of these relationships; so it is not part of the *range* {sweetness, tartness}.

Another way of looking at this is to say that a relation is a *subset of ordered pairs* drawn from the *set of all possible ordered pairs* (of elements of two other sets, which we normally refer to as the *Cartesian product* of those sets). Formally, R is a relation if

R ⊆ {(x, y) | x ∈ X, y ∈ Y}

for the domain X and codomain Y.

Using the example above, we can write the relation in set notation: {(apples, sweetness), (apples, tartness), (oranges, tartness), (bananas, sweetness)}.

One important kind of relation is the *function*. A ***function*** is a relation that has **exactly one output** for every possible input **in the domain**. (Unlike the codomain, the domain does not necessarily have to include all possible objects of a given type. In fact, we sometimes intentionally use a *restricted domain* in order to satisfy some desirable property.) For example, the relation that we discussed above (flavors of fruits) is **not** a function, because it has two possible outputs for the input "apples": sweetness and tartness.

The main reason for not allowing multiple outputs with the same input is that it lets us apply the same function to different forms of the same thing without changing their equivalence. That is, if x = y, and f is a function with x (or y) in its domain, then f(x) = f(y). For example, z - 3 = 5 implies that z = 8 because f(x) = x + 3 is a function defined for all numbers x.

The converse, that f(x) = f(y) implies x = y, is not always true. When it is, f is called a **one-to-one** or **invertible function**.

## Relations

In the above section dealing with functions and their properties, we noted the important property that all functions must have, namely that if a function does map a value from its domain to its co-domain, it must map this value to only one value in the co-domain.

Writing in set notation, if *a* is some fixed value:

|{f(x)|x=a}| ∈ {0, 1}

The literal reading of this statement is: the *cardinality* (number of elements) of the set of all values f(x), such that x=a for some fixed value a, is an element of the set {0, 1}. In other words, the number of *outputs* that a function f may have at any fixed *input* a is either zero (in which case it is *undefined* at that input) or one (in which case the output is unique).

However, when we consider the *relation*, we relax this constriction, and so a relation may map one value to more than one other value. In general, a relation is **any** subset of the Cartesian product of its domain and co-domain.

All functions, then, can be considered as relations also.

### Notations

When we have the property that one value is related to another, we call this relation a *binary relation* and we write it as

x R y

where R is the relation.

For arrow diagrams and set notations, remember for relations we do not have the restriction that functions do and we can draw an arrow to represent the mappings, and for a set diagram, we need only write all the ordered pairs that the relation does take: again, by example

f = {(0,0),(1,1),(1,-1),(2,2),(2,-2)}

is a relation and not a function, since both 1 and 2 are mapped to two values, 1 and -1, and 2 and -2 respectively) example let A=2,3,5;B=4,6,9 then A\*B=(2,4),(2,6),(2,9),(3,4),(3,6),(3,9),(5,4),(5,6),(5,9) Define a relation R=(2,4),(2,6),(3,6),(3,9) add funtions and problems to one aonther

### Some simple examples

Let us examine some simple relations.

Say f is defined by

{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(3,1),(2,1),(3,2),(1,3)}

This is a relation (not a function) since we can observe that 1 maps to 2 and 3, for instance.

Less-than, "<", is a relation also. Many numbers can be less than some other fixed number, so it cannot be a function.

### Properties

When we are looking at relations, we can observe some special properties different relations can have.

#### Reflexive

A relation is *reflexive* if, we observe that for all values a:

*a* R *a*

In other words, all values are related to themselves.

The relation of equality, "=" is reflexive. Observe that for, say, all numbers a (the domain is **R**):

*a* = *a*

so "=" is reflexive.

In a reflexive relation, we have arrows for all values in the domain pointing back to themselves:

[Arrow diagram reflexive.png](http://commons.wikimedia.org/wiki/File:Arrow_diagram_reflexive.png)

Note that ≤ is also reflexive (a ≤ a for any a in **R**). On the other hand, the relation < is not (a < a is false for any a in **R**).

#### Symmetric

A relation is *symmetric* if, we observe that for all values a and b:

*a* R *b* implies *b* R *a*

The relation of equality again is symmetric. If *x*=*y*, we can also write that *y*=*x* also.

In a symmetric relation, for each arrow we have also an opposite arrow, i.e. there is either no arrow between *x* and *y*, or an arrow points from *x* to *y* and an arrow back from *y* to *x*:

[Arrow diagram symmetric.png](http://commons.wikimedia.org/wiki/File:Arrow_diagram_symmetric.png)

Neither ≤ nor < is symmetric (2 ≤ 3 and 2 < 3 but not 3 ≤ 2 nor 3 < 2 is true).

#### Transitive

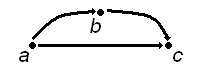
A relation is *transitive* if for all values *a*, *b*, *c*:

*a* R *b* and *b* R *c* implies *a* R *c*

The relation *greater-than* ">" is transitive. If *x* > *y*, and *y* > *z*, then it is true that *x* > *z*. This becomes clearer when we write down what is happening into words. *x* is greater than *y* and *y* is greater than *z*. So *x* is greater than both *y* and *z*.

The relation *is-not-equal* "≠" is not transitive. If *x* ≠ *y* and *y* ≠ *z* then we might have *x* = *z* or *x* ≠ *z* (for example 1 ≠ 2 and 2 ≠ 3 and 1 ≠ 3 but 0 ≠ 1 and 1 ≠ 0 and 0 = 0).

In the arrow diagram, every arrow between two values *a* and *b*, and *b* and *c*, has an arrow going straight from *a* to *c*.

[](http://commons.wikimedia.org/wiki/File:Arrow_diagram_transitive.png)

#### Antisymmetric

A relation is *antisymmetric* if we observe that for all values *a* and *b*:

*a* R *b* and *b* R *a* implies that *a*=*b*

**Notice that antisymmetric is not the same as "not symmetric."**

Take the relation *greater than equals to*, "≥" If *x* ≥ *y*, and *y* ≥ x, then *y* must be equal to *x*. a relation is anti-symmetric if and only if a∈A, (a,a)∈R

#### Trichotomy

A relation satisfies *trichotomy* if we observe that for all values *a* and *b* it holds true that: *a*R*b* *or* *b*R*a*

The relation *is-greater-or-equal* satisfies since, given 2 real numbers *a* and *b*, it is true that whether *a* ≥ *b* or *b* ≥ *a* (both if *a* = *b*).

#### Problem set

Given the above information, determine which relations are reflexive, transitive, symmetric, or antisymmetric on the following - there may be more than one characteristic. (Answers follow.) *x* R *y* if

1. x = y
2. x < y
3. x2 = y2
4. x ≤ y

##### Answers

1. Symmetric, Reflexive, Transitive and Antisymmetric
2. Transitive
3. Symmetric, Reflexive, Transitive and Antisymmetric (x2 = y2 is just a special case of equality, so all properties that apply to x = y also apply to this case)
4. Reflexive, Transitive and Antisymmetric (and satisfying Trichotomy)

### Equivalence relations

We have seen that certain common relations such as "=", and congruence (which we will deal with in the next section) obey some of these rules above. The relations we will deal with are very important in discrete mathematics, and are known as *equivalence relations*. They essentially assert some kind of equality notion, or *equivalence*, hence the name.

#### Characteristics of equivalence relations

For a relation R to be an *equivalence relation*, it must have the following properties, viz. R must be:

* symmetric
* transitive
* reflexive

(A helpful mnemonic, S-T-R)

In the previous problem set you have shown equality, "=", to be reflexive, symmetric, and transitive. So "=" is an equivalence relation.

We denote an equivalence relation, in general, by x \sim y.

#### Example proof

Say we are asked to prove that "=" is an equivalence relation. We then proceed to prove each property above in turn (Often, the proof of transitivity is the hardest).

##### Reflexive

Clearly, it is true that *a* = *a* for all values a. So = is reflexive.

##### Symmetric

If *a* = *b*, it is also true that *b* = *a*. So = is symmetric

##### Transitive

If *a* = *b* and *b* = *c*, this says that *a* is the same as *b* which in turn is the same as *c*. So *a* is then the same as *c*, so *a* = *c*, and thus = is transitive.

Thus = is an equivalence relation.

any relation R on set A is said to be a transitive relation.

(a,b),(b,c)belongs to(a,c)belongs to R,where a,b,c belongs to a.

#### Partitions and equivalence classes

It is true that when we are dealing with relations, we may find that many values are related to one fixed value.

For example, when we look at the quality of *congruence*, which is that given some number *a*, a number congruent to *a* is one that has the same remainder or *modulus* when divided by some number *n*, as *a*, which we write

a ≡ b (mod n)

and is the same as writing

*b* = *a*+*kn* for some integer k.

(We will look into congruences in further detail later, but a simple examination or understanding of this idea will be interesting in its application to equivalence relations)

For example, 2 ≡ 0 (mod 2), since the remainder on dividing 2 by 2 is in fact 0, as is the remainder on dividing 0 by 2.

We can show that congruence is an equivalence relation (This is left as an exercise, below **Hint** use the equivalent form of congruence as described above).

However, what is more interesting is that we can group all numbers that are equivalent to each other.

With the relation congruence *modulo* 2 (which is using n=2, as above), or more formally:

x ~ y if and only if x ≡ y (mod 2)

we can group all numbers that are equivalent to each other. Observe:

0 \equiv 2 \equiv 4 \equiv \ldots \pmod{2}

1 \equiv 3 \equiv 5 \equiv \ldots \pmod{2}

This first equation above tells us all the *even* numbers are equivalent to each other under ~, and all the *odd* numbers under ~.

We can write this in set notation. However, we have a special notation. We write:

[0]={0,2,4,...}

[1]={1,3,5,...}

and we call these two sets *equivalence classes*.

All elements in an equivalence class by definition are equivalent to each other, and thus note that we do not need to include [2], since 2 ~ 0.

We call the act of doing this 'grouping' with respect to some equivalence relation *partitioning* (or further and explicitly *partitioning a set S into equivalence classes under a relation ~*). Above, we have partitioned **Z** into equivalence classes [0] and [1], under the relation of congruence modulo 2.

#### Problem set

Given the above, answer the following questions on equivalence relations (Answers follow to even numbered questions)

1. Prove that congruence is an equivalence relation as before (See hint above).
2. Partition {x | 1 ≤ x ≤ 9} into equivalence classes under the equivalence relation

 x \sim y\ \mbox{iff}\ x \equiv y \pmod{6}

##### Answers

2. [0]={6}, [1]={1,7}, [2]={2,8}, [3]={3,9}, [4]={4}, [5]={5}

### Partial orders

We also see that "≥" and "≤" obey some of the rules above. Are these special kinds of relations too, like equivalence relations? Yes, in fact, these relations are specific examples of another special kind of relation which we will describe in this section: the *partial order*.

As the name suggests, this relation gives some kind of ordering to numbers.

#### Characteristics of partial orders

For a relation R to be a partial order, it must have the following three properties, viz R must be:

* reflexive
* antisymmetric
* transitive

(A helpful mnemonic, R-A-T)

We denote a partial order, in general, by x \preceq y.

#### Example proof

Say we are asked to prove that "≤" is a partial order. We then proceed to prove each property above in turn (Often, the proof of transitivity is the hardest).

##### Reflexive

Clearly, it is true that *a* ≤ *a* for all values a. So ≤ is reflexive.

##### Antisymmetric

If *a* ≤ *b*, and *b* ≤ *a*, then a *must* be equal to *b*. So ≤ is antisymmetric

##### Transitive

If *a* ≤ *b* and *b* ≤ *c*, this says that *a* is less than *b* and *c*. So *a* is less than *c*, so *a* ≤ *c*, and thus ≤ is transitive.

Thus ≤ is a partial order.

#### Problem set

Given the above on partial orders, answer the following questions

1. Prove that divisibility, |, is a partial order (a | b means that a is a factor of b, i.e., on dividing b by a, no remainder results).
2. Prove the following set is a partial order: (*a*, *b*) \preceq(*c*, *d*) implies *ab*≤*cd* for *a*,*b*,*c*,*d* integers ranging from 0 to 5.

##### Answers

2. Simple proof; Formalization of the proof is an optional exercise.

Reflexivity: (*a*, *b*) \preceq(*a*, *b*) since *ab*=*ab*.

Antisymmetric: (*a*, *b*) \preceq(*c*, *d*) and (*c*, *d*) \preceq(*a*, *b*) since *ab*≤*cd* and *cd*≤*ab* imply *ab*=*cd*.

Transitive: (*a*, *b*) \preceq(*c*, *d*) and (*c*, *d*) \preceq(*e*, *f*) implies (*a*, *b*) \preceq(*e*, *f*) since *ab*≤*cd*≤*ef* and thus *ab*≤*ef*

#### Posets

A partial order imparts some kind of "ordering" amongst elements of a set. For example, we only know that 2 ≥ 1 because of the partial ordering ≥.

We call a set A, ordered under a general partial ordering \preceq, a *partially ordered set*, or simply just *poset*, and write it (A, \preceq).

##### Terminology

There is some specific terminology that will help us understand and visualize the partial orders.

When we have a partial order \preceq, such that *a* \preceq*b*, we write \precto say that a \preceqbut *a* ≠ *b*. We say in this instance that a *precedes* b, or *a* is a predecessor of *b*.

If (A, \preceq) is a poset, we say that *a* is an immediate predecessor of *b* (or *a* immediately precedes *b*) if there is no *x* in A such that *a* \prec*x* \prec*b*.

If we have the same poset, and we also have *a* and *b* in A, then we say *a* and *b* are *comparable* if *a* \preceq*b* and *b* \preceq*a*. Otherwise they are *incomparable*.

#### Hasse diagrams

*Hasse diagrams* are special diagrams that enable us to visualize the structure of a partial ordering. They use some of the concepts in the previous section to draw the diagram.

A Hasse diagram of the poset (A, \preceq) is constructed by

* placing elements of A as points
* if *a* and *b* ∈ A, and *a* is an immediate predecessor of b, we draw a line from *a* to *b*
* if *a* \prec*b*, put the point for *a* lower than the point for *b*
* not drawing loops from *a* to *a* (this is assumed in a partial order because of reflexivity)